

# ON EXTREMAL PROPERTIES OF JACOBIAN ELLIPTIC FUNCTIONS WITH COMPLEX MODULUS

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**ABSTRACT.** A thorough analysis of values of the function  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  for complex parameter  $m$  and  $u \in (0, 1)$  is given. First, it is proved that the absolute value of this function never exceeds 1 if  $m$  does not belong to the region in  $\mathbb{C}$  determined by inequalities  $|z - 1| < 1$  and  $|z| > 1$ . The global maximum of the function under investigation is shown to be always located in this region. More precisely, it is proved that, if  $u \leq 1/2$ , then the global maxim is located at  $m = 1$  with the value equal to 1. While if  $u > 1/2$ , then the global maximum is located in the interval  $(1, 2)$  and its value exceeds 1. In addition, more subtle extremal properties are studied numerically. Finally, applications in a Laplace-type integral and spectral analysis of some complex Jacobi matrices are presented.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Jacobian elliptic functions  $\operatorname{sn}(z \mid m)$ ,  $\operatorname{cn}(z \mid m)$ ,  $\operatorname{dn}(z \mid m)$  have been studied in depth, in particular, as functions of complex argument  $z$ ; classical references are treatises [2, 8, 11, 18] or handbooks [1, 4]. However, the vast majority of investigations have been made when the parameter  $m$  (or modulus  $k$ ; see Section 2.1 for details) is real and restricted to the interval  $(0, 1)$ , which typically suffices in applications. The articles of Walker [16, 17] and Schiefermayr [13] belong to the few exceptions devoted directly to the study of Jacobian elliptic functions as functions of complex parameter  $m$ .

The aim of this paper is an investigation of properties of the function  $m \mapsto \operatorname{sn}(K(m)u \mid m)$ , for  $u \in \mathbb{R}$  and  $m \in \mathbb{C}$ , where  $K(m)$  stands for the complete elliptic integral of the first kind. Mainly, we focus on the localization of range of the function  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$ , and particularly, its extremal values.

The composition of the function  $\operatorname{sn}$  with the complete elliptic integral  $K$  is quite natural as one can observe, for example, from the relation with Jacobi's theta functions, see (5) below, or Eisenstein series [12]; see also the discussion in [7] and the applications in Section 5.

While  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  is an analytic function in the cut-plane  $\mathbb{C} \setminus [1, \infty)$  with the branch cut in  $[1, \infty)$ , its modulus is well defined, bounded and continuous function in the whole complex plane  $\mathbb{C}$ . Our main result is the following theorem reflecting the non-trivial properties of  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$ .

**Theorem 1.** *The following statements hold true.*

- i) *For all  $u \in (0, 1)$  and  $m \notin \{z \in \mathbb{C} : |z - 1| < 1 \wedge |z| > 1\}$ , it holds*

$$|\operatorname{sn}(K(m)u \mid m)| < 1. \quad (1)$$

- ii) *For  $u \in (0, 1/2]$  the function  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  has unique global maximum located at  $m = 1$  with the value equal to 1, i.e.,*

$$|\operatorname{sn}(K(1)u \mid 1)| = 1 \quad \text{and} \quad |\operatorname{sn}(K(m)u \mid m)| < 1 \quad \text{for all } m \neq 1$$

*(where the value at  $m = 1$  is to be understood as the respective limit).*

- iii) *For  $u \in (1/2, 1)$ , the function  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  has a global maximum located in the interval  $(1, 2)$  with the value exceeding 1, i.e.,*

$$\max_{m \in \mathbb{C}} |\operatorname{sn}(K(m)u \mid m)| = |\operatorname{sn}(K(m^*)u \mid m^*)| > 1 \quad \text{for some } m^* \in (1, 2).$$

*Date:* December 16, 2015.

*2010 Mathematics Subject Classification.* 33E05.

*Key words and phrases.* Jacobian elliptic functions, complex modulus, extrema of elliptic functions.

Notice that there is no loss of generality in restricting  $u$  to  $(0, 1)$ , which is assumed throughout the whole paper, since the function  $u \mapsto |\operatorname{sn}(K(m)u | m)|$  is 2-periodic,  $\operatorname{sn}(K(m)(2-u) | m) = \operatorname{sn}(K(m)u | m)$  and  $\operatorname{sn}(0 | m) = 0$ ,  $\operatorname{sn}(K(m) | m) = 1$ .

The proof of Theorem 1, consisting of several steps, is presented in Section 3. Besides the employment of many known properties (transformation and addition rules, derivatives w.r.t. to both argument and parameter, asymptotic expansions as  $m \rightarrow 0$  and  $m \rightarrow 1$ ) of Jacobian elliptic functions, it heavily relies on the maximum modulus principle both for bounded and unbounded regions.

The situation for  $m \in \{z \in \mathbb{C} : |z - 1| < 1 \wedge |z| > 1\}$  is more delicate and the region where (1) holds seems to have an interesting shape, see Figure 1. Moreover, the value of the global maximum of  $|\operatorname{sn}(K(m)u | m)|$  on  $(u, m) \in \mathbb{R} \times \mathbb{C}$  seems to exceed 1 only slightly by approx. 0.01. For more details and other numerical results, see Section 4.

Finally, in Section 5, we explain an application of Theorem 1 in the spectral analysis of certain family of semi-infinite complex Jacobi matrices whose diagonal vanishes and off-diagonal is an unbounded periodically modulated sequence. In this case, the coupling constant presented in the modulated weight coincides with the modulus of a Jacobian elliptic function. The spectral problem for the corresponding family of orthogonal polynomials, with modulus being restricted to  $(0, 1)$ , has been studied by Carlitz in [5, 6]. His motivation was, in turn, based on beautiful continued fraction formulas for Laplace transform of powers of Jacobian elliptic functions, which goes back to Stieltjes [15]. Theorem 1 allows to treat the spectral analysis for general complex parameter; details are elaborated in a subsequent paper [14].

## 2. PRELIMINARIES

**2.1. Notations, auxiliary functions and analyticity properties.** As all the Jacobian elliptic functions depend on  $k^2$  rather than the modulus  $k$  itself, we primarily follow Milne-Thompson's notation using parameter  $m$  instead of the modulus  $k$ , i.e.  $m = k^2$  and  $\operatorname{sn}(z | m) = \operatorname{sn}(z, k)$ , etc; this is also the case for Abramowitz & Stegun [1, Chap. 16] as well as [16, 17]. In the classical theory, the modulus  $k$  is accompanied with the complementary modulus  $k'$  which are related by equation  $k^2 + k'^2 = 1$ ; in terms of parameter we use  $m$  and  $m_1 := 1 - m$ .

We use the following notation for the upper and lower complex half-plane

$$\mathbb{C}_{\pm} := \{z \in \mathbb{C} \mid \operatorname{Im} z \gtrless 0\},$$

and the subsets of  $\mathbb{C}$  related to the unit disks centered and 0 and 1 are denoted by

$$\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{D}_1 := \{z \in \mathbb{C}, |z - 1| < 1\}, \quad \overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}, \quad \overline{\mathbb{D}}_1 := \mathbb{D}_1 \cup \partial\mathbb{D}_1.$$

Further, throughout the whole paper, the principal part of the square root is always used.

Recall the complete elliptic integral of the first kind is defined by

$$K(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}$$

and it is analytic for  $m \in \mathbb{C} \setminus [1, \infty)$ . If not necessary, we will suppress the dependence on  $m$  in the notation of  $K$ .  $K(m)$  has a singularity in  $m = 1$  and a branch cut in  $(1, \infty)$ . Limit values of  $K(m)$ , for  $m > 1$ , exist if  $m$  is approached from one of the half-planes  $\mathbb{C}_+$  or  $\mathbb{C}_-$ . More precisely, one has the connection formula

$$K(1/m) = m^{1/2} [K(m) \mp iK'(m)], \quad (2)$$

where  $K'(m) = K(m_1)$  and the upper or lower sign is taken according as  $m \in \mathbb{C}_+$  or  $m \in \mathbb{C}_-$ , respectively. For references and an interesting discussion of this dichotomy in sign, see [10].

In the theory of elliptic functions, a conformal mapping known as the elliptic modular function  $\lambda : \tau \mapsto m$  is investigated. For the inverse mapping, one has

$$\tau = i \frac{K'(m)}{K(m)}. \quad (3)$$

The complementary variable to  $\tau$  is denoted here by  $\tau_1$  where  $\tau_1 = -1/\tau$ . In addition, the nome  $q$  is given by

$$q = e^{i\pi\tau} = e^{-\pi \frac{K'(m)}{K(m)}}. \quad (4)$$

Properties of the function  $\lambda$  are essential for understanding of dependencies and analyticity of mappings relating quantities  $m \leftrightarrow \tau \leftrightarrow q$ , see [8, Chap. 7, §8].

Further recall that all Jacobian elliptic functions can be defined as a quotient of Jacobi's theta functions, see [11, Chap. 2]. For the function  $\text{sn}$ , one has

$$\text{sn}(K(m)u \mid m) = \frac{\theta_3(0, q) \theta_1(u\pi/2, q)}{\theta_2(0, q) \theta_4(u\pi/2, q)}. \quad (5)$$

All four theta functions are analytic functions in  $m$  for  $m \in \mathbb{C} \setminus [1, \infty)$ , see [16, Sec. 4]. Since all zeros of the theta functions are known explicitly, see [11, Eq. 1.3.10], it can be easily verified that theta functions in the denominator of the RHS in (5) never vanish for  $u \in \mathbb{R}$  and  $m \in \mathbb{C} \setminus [1, \infty)$ . Consequently, for  $u \in \mathbb{R}$ , the function  $m \mapsto \text{sn}(K(m)u \mid m)$  is analytic in the cut-plane  $\mathbb{C} \setminus [1, \infty)$ .

Moreover, since  $q(\bar{m}) = \overline{q(m)}$  and  $\theta_l(x, \bar{q}) = \overline{\theta_l(x, q)}$ , for  $x \in \mathbb{R}$  and  $l \in \{1, 2, 3, 4\}$ , it follows from (5) that, for  $u \in \mathbb{R}$  and  $m \in \mathbb{C} \setminus [1, \infty)$ ,

$$\overline{\text{sn}(K(m)u \mid m)} = \text{sn}(K(\bar{m})u \mid \bar{m}). \quad (6)$$

**2.2. Asymptotic expansions for  $m$  close to 0 and 1.** For later purposes, we first derive asymptotic expansions of  $\text{sn}(K(m)u \mid m)$ ,  $\text{cn}(K(m)u \mid m)$  and  $\text{dn}(K(m)u \mid m)$  for  $m$  close to 0 and 1.

Restricted to  $m \in (0, 1)$ , Carlson and Todd proved in [7] that, for  $u \in (0, 1)$ , the function  $m \mapsto \text{sn}(K(m)u \mid m)$  is strictly increasing on  $(0, 1)$ , cf. [7, Thm. 1], and found its asymptotic behavior as  $m \rightarrow 0+$  and  $m \rightarrow 1-$ , cf. [7, Sec. 4, Thm. 3].

We require  $m$  to be confined to the set  $\mathbb{C} \setminus [1, \infty)$  only. While the proof for  $m \rightarrow 0$  follows the usual strategy and straightforwardly reduces to known facts, the case when  $m \rightarrow 1$  is a bit more difficult and we present a method based of the Fourier series expansions of some Jacobian elliptic functions.

**Lemma 2.** *For  $u \in (0, 1)$ , we have the following asymptotic expansions*

$$\text{sn}(K(m)u \mid m) = \sin(\pi u/2) + O(m), \quad (7)$$

$$\text{cn}(K(m)u \mid m) = \cos(\pi u/2) + O(m), \quad (8)$$

$$\text{dn}(K(m)u \mid m) = 1 + O(m), \quad (9)$$

as  $m \rightarrow 0$ ,  $m \in \mathbb{C} \setminus [1, \infty)$  and

$$\text{sn}(K(m)u \mid m) = 1 - 2^{1-4u} m_1^u + o(m_1^u), \quad (10)$$

$$\text{cn}(K(m)u \mid m) = 2^{1-2u} m_1^{u/2} + o(m_1^{u/2}), \quad (11)$$

$$\text{dn}(K(m)u \mid m) = 2^{1-2u} m_1^{u/2} + o(m_1^{u/2}), \quad (12)$$

as  $m \rightarrow 1$ ,  $m \in \mathbb{C} \setminus [1, \infty)$ .

*Proof.* The case  $m \rightarrow 0$ : It follows that  $q \rightarrow 0$  and one can make use of the relation between the three functions and Jacobi's theta functions [11, Eqs. 2.1.1–3] together with the well known expansions of the theta functions for  $q \rightarrow 0$ , see [11, Eqs. 2.1.14–17]. This yields

$$\text{sn}(K(m)u \mid m) = \sin(\pi u/2) + O(q),$$

$$\text{cn}(K(m)u \mid m) = \cos(\pi u/2) + O(q),$$

$$\text{dn}(K(m)u \mid m) = 1 + O(q),$$

for  $m \rightarrow 0$ . Of course, it is desirable to rewrite the RHSs in the last expansions in terms of the modulus  $m$ . To do this, recall [11, Eq. 2.1.12]

$$m = 16q \prod_{n=1}^{\infty} \left( \frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8,$$

from which one deduces

$$q = \frac{1}{16}m + \frac{1}{32}m^2 + O(m^3), \quad m \rightarrow 0, \quad (13)$$

thus we arrive at the expansions (7)–(9) in the claim.

The case  $m \rightarrow 1$ : First, recall Fourier series expansions for functions  $\text{sc}$ ,  $\text{nc}$ , and  $\text{dc}$ , see [1, Eqs. 16.23.7–9], valid whenever  $|\text{Im } z/K| < 2 \text{Im } \tau$ , see [11, Sec. 8.7],

$$\text{sc}(z | m) = \frac{\pi}{2m_1^{1/2}K} \tan\left(\frac{\pi z}{2K}\right) + \frac{2\pi}{m_1^{1/2}K} \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{1+q^{2n}} \sin\left(\frac{n\pi z}{K}\right), \quad (14)$$

$$\text{nc}(z | m) = \frac{\pi}{2m_1^{1/2}K} \sec\left(\frac{\pi z}{2K}\right) - \frac{2\pi}{m_1^{1/2}K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1+q^{2n+1}} \cos\left(\frac{(2n+1)\pi z}{2K}\right), \quad (15)$$

$$\text{dc}(z | m) = \frac{\pi}{2K} \sec\left(\frac{\pi z}{2K}\right) + \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1-q^{2n+1}} \cos\left(\frac{(2n+1)\pi z}{2K}\right). \quad (16)$$

Writing the trigonometric functions in (14)–(16) in terms of exponentials and using (4), we arrive at formulas that are more convenient for our purposes:

$$\text{sc}(\tau Ku | m) = \frac{i\pi}{2m_1^{1/2}K} \frac{1-q^u}{1+q^u} + \frac{i\pi}{m_1^{1/2}K} \sum_{n=1}^{\infty} (-1)^n \frac{q^{(2-u)n}}{1+q^n} (1-q^{2nu}), \quad (17)$$

$$\text{nc}(\tau Ku | m) = \frac{\pi}{m_1^{1/2}K} \frac{q^{u/2}}{1+q^u} - \frac{\pi}{m_1^{1/2}K} \sum_{n=0}^{\infty} (-1)^n \frac{q^{(n+1/2)(2-u)}}{1+q^{2n+1}} (1+q^{(2n+1)u}), \quad (18)$$

$$\text{dc}(\tau Ku | m) = \frac{\pi}{K} \frac{q^{u/2}}{1+q^u} + \frac{\pi}{K} \sum_{n=0}^{\infty} (-1)^n \frac{q^{(n+1/2)(2-u)}}{1-q^{2n+1}} (1+q^{(2n+1)u}), \quad (19)$$

where  $u \in (-2, 2)$ . If  $m \rightarrow 0$ , then  $q \rightarrow 0$ , thus, from equalities (17)–(19), we immediately deduce the asymptotic expansions formulas for  $m \rightarrow 0$  in terms of  $q$  and subsequently, taking into account (13) and noticing that  $K(m) \rightarrow \pi/2$  for  $m \rightarrow 0$ , we obtain the asymptotic expansions

$$\text{sc}(\tau Ku | m) = i - i2^{1-4u}m^u + o(m^u), \quad (20)$$

$$\text{nc}(\tau Ku | m) = 2^{1-2u}m^{u/2} + o(m^{u/2}), \quad (21)$$

$$\text{dc}(\tau Ku | m) = 2^{1-2u}m^{u/2} + o(m^{u/2}), \quad (22)$$

for  $m \rightarrow 0$  and  $u \in (0, 1)$ .

Finally, by the Jacobi's imaginary transformation [1, Eq. 16.20.1], one gets

$$\text{sn}(K(m)u | m) = -i \text{sc}(iK(m)u | m_1) = -i \text{sc}(\tau_1 K(m_1)u | m_1)$$

where the last equality holds since for the complementary variable  $\tau_1$  one has  $\tau_1 = -1/\tau = iK(m)/K(m_1)$ . Note that if  $m \rightarrow 1$ , then  $m_1 \rightarrow 0$ , thus to obtain (10), it suffices to apply (20). The remaining two expansions for  $\text{cn}(K(m)u | m)$  and  $\text{dn}(K(m)u | m)$  follow in an analogous way; the corresponding Jacobi's imaginary transformation [1, Eqs. 16.20.2–3] and expansions (21)–(22) are used.  $\square$

### 3. PROOF OF THE MAIN RESULT

**3.1. Region  $m \in \overline{\mathbb{D}}$ .** We begin with analysis of  $|\text{sn}(K(m)u | m)|$  for  $m$  in  $\partial\mathbb{D} \setminus \{1\}$  and subsequently extend the obtained inequality on  $\overline{\mathbb{D}} \setminus \{1\}$ .

**Lemma 3.** *For all  $(u, m) \in (0, 1) \times \partial\mathbb{D} \setminus \{1\}$ , we have*

$$|\text{sn}(K(m)u | m)| < 1.$$

*Proof.* Taking into account (6), it is sufficient to verify the statement for  $m = e^{4i\theta}$  with  $\theta \in (0, \pi/4]$ . First, we apply the ascending Landen transformation [1, Eq. 16.14.2] on  $\text{sn}(K(m)u | m)$  and receive the equality

$$\text{sn}(K(e^{4i\theta})u | e^{4i\theta}) = \frac{e^{-i\theta}}{\cos \theta} \frac{\text{sn}(K(e^{4i\theta})v | \cos^{-2}\theta) \text{cn}(K(e^{4i\theta})v | \cos^{-2}\theta)}{\text{dn}(K(e^{4i\theta})v | \cos^{-2}\theta)},$$

where  $v = ue^{i\theta} \cos \theta$ . Next, by the double argument formula [1, Eq. 16.18.5], we arrive at

$$\text{sn}^2(K(e^{4i\theta})u | e^{4i\theta}) = e^{-2i\theta} \frac{1 - \text{dn}(2K(e^{4i\theta})v | \cos^{-2}\theta)}{1 + \text{dn}(2K(e^{4i\theta})v | \cos^{-2}\theta)}$$

and by the Jacobi's real transformation [1, Eq. 16.11.4], we get

$$\operatorname{sn}^2(K(e^{4i\theta})u \mid e^{4i\theta}) = e^{-2i\theta} \frac{1 - \operatorname{cn}(2K(e^{4i\theta})e^{i\theta}u \mid \cos^2\theta)}{1 + \operatorname{cn}(2K(e^{4i\theta})e^{i\theta}u \mid \cos^2\theta)}.$$

Finally, the application of formula

$$K(e^{4i\theta}) = \frac{1}{2}e^{-i\theta} [K(\cos^2\theta) + iK(\sin^2\theta)], \quad \text{for } \theta \in (0, \pi/4], \quad (23)$$

see [10] and references therein, yields

$$\operatorname{sn}^2(K(e^{4i\theta})u \mid e^{4i\theta}) = e^{-2i\theta} \frac{1 - \operatorname{cn}([K(\cos^2\theta) + iK(\sin^2\theta)]u \mid \cos^2\theta)}{1 + \operatorname{cn}([K(\cos^2\theta) + iK(\sin^2\theta)]u \mid \cos^2\theta)} \quad (24)$$

where  $\theta \in (0, \pi/4]$ . By [1, Eq. 16.21.3], we have

$$\operatorname{cn}([K(\cos^2\theta) + iK(\sin^2\theta)]u \mid \cos^2\theta) = \frac{c \cdot c_1 - is \cdot s_1 \cdot d \cdot d_1}{c_1^2 + \cos^2(\theta) s^2 \cdot s_1^2}$$

where the abbreviations  $s = \operatorname{sn}(K(\cos^2\theta)u \mid \cos^2\theta)$ ,  $s_1 = \operatorname{sn}(K(\sin^2\theta)u \mid \sin^2\theta)$ , etc., are used. With the aid of the last formula and identity (24) one deduces that

$$|\operatorname{sn}(K(e^{4i\theta})u \mid e^{4i\theta})| < 1$$

if and only if  $(c_1^2 + \cos^2(\theta) s^2 \cdot s_1^2) c \cdot c_1 > 0$ . However, the latter inequality is clearly true since all the functions  $s$ ,  $s_1$ ,  $c$ ,  $c_1$  are positive for  $u \in (0, 1)$ .  $\square$

**Proposition 4.** *For all  $(u, m) \in (0, 1) \times \overline{\mathbb{D}} \setminus \{1\}$ , we have*

$$|\operatorname{sn}(K(m)u \mid m)| < 1. \quad (25)$$

*Proof.* By Lemma 3, the statement holds for  $m \in \partial\mathbb{D} \setminus \{1\}$ , hence it suffices to verify the inequality for  $m \in \mathbb{D}$ . Note that by (10),

$$\lim_{\substack{m \rightarrow 1 \\ m \in \mathbb{D} \setminus \{1\}}} \operatorname{sn}(K(m)u \mid m) = 1.$$

Thus, the function  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  extends continuously to  $\overline{\mathbb{D}}$ .

For  $m \in \mathbb{D}$ , we denote  $f(m) := \operatorname{sn}(K(m)u \mid m)$ . Recall that the Möbius transform

$$\varphi_m(z) = \frac{m - z}{1 - \overline{m}z}$$

is a conformal self-map of the unit disk  $\mathbb{D}$ . Moreover,  $\varphi_m$  maps the boundary  $\partial\mathbb{D}$  into itself. Hence  $g := f \circ \varphi_m$  is analytic on  $\mathbb{D}$ , continuous on the closure  $\overline{\mathbb{D}}$ , and  $g(0) = f(m)$ . By the Cauchy integral formula, one has

$$|f(m)| = |g(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})| d\theta,$$

for any  $r \in (0, 1)$ . By the Lebesgue's dominated convergence theorem, one can send  $r \rightarrow 1-$  in the above integral and thereby obtain the inequality

$$|f(m)| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})| d\theta.$$

The statement follows, since by Lemma 3,  $|g(e^{i\theta})| < 1$  for almost all  $\theta \in (0, 2\pi)$  and hence

$$\int_0^{2\pi} |g(e^{i\theta})| d\theta < 2\pi. \quad \square$$

**3.2. Region**  $m \in \partial\mathbb{D}_1$ . Here we show the values of  $|\operatorname{sn}(K(m)u \mid m)|$  remain below 1 also for  $m$  on the boundary of  $\mathbb{D}_1$ . For this purpose, we need the following auxiliary inequality.

**Lemma 5.** *For all  $(x, m) \in (0, 2K) \times [0, 1/2]$ , it holds*

$$4 \operatorname{dn}^2(x \mid m) (1 + \operatorname{cn}(x \mid m)) - \operatorname{sn}^2(x \mid m) (1 - \operatorname{cn}(x \mid m)) > 0. \quad (26)$$

*Proof.* Denote the function on the LHS of inequality (26) by  $f(x)$ . We show that  $f$  is strictly decreasing on  $(0, 2K)$  and since  $f(2K) = 0$ , the statement will be proved.

We abbreviate  $s := \operatorname{sn}(x \mid m)$ , etc. By using the formulas for the derivatives of Jacobian elliptic functions, see [1, Eqs. 16.16.1–3], and elementary identities  $d^2 = 1 - m + mc^2$ ,  $s^2 = 1 - c^2$ , one computes  $f'(x) = -s \cdot d \cdot g(x)$  where

$$g(x) = 2(4m + 1)c + 2(4m - 1)c^2 + 4d^2 + s^2 = 5 - 4m + 2(4m + 1)c + 3(4m - 1)c^2.$$

Since  $s \cdot d > 0$  on  $(0, 2K)$ , it suffices to show that  $g(x) > 0$  for  $x \in (0, 2K)$ . We distinguish to cases when  $x \in (0, K]$  and  $x \in (K, 2K)$ .

i) Let  $x \in (0, K]$ . Since

$$2(4m + 1)c \geq 2c \quad \text{and} \quad 3(4m - 1)c^2 \geq -3c^2 \geq -3c,$$

one gets  $g(x) \geq 4(1 - m) + 1 - c > 0$  for all  $m \in [0, 1]$ .

ii) Let  $x \in (K, 2K)$ . We show that  $g$  is strictly decreasing on  $(K, 2K)$  which together with identity  $g(2K) = 0$  implies positivity of  $g$ . Differentiating  $g$  yields

$$g'(x) = -2s \cdot d [4m + 1 + 3(4m - 1)c].$$

Since  $c > -1$ , the expression in the square brackets in the last formula can be estimated as

$$4m + 1 + 3(4m - 1)c > 4(1 - 2m),$$

hence  $g'(x) < 0$  for all  $x \in (K, 2K)$  and  $m \in [0, 1/2]$ .  $\square$

**Lemma 6.** *For all  $(u, m) \in (0, 1) \times \partial\mathbb{D}_1$ , we have*

$$|\operatorname{sn}(K(m)u \mid m)| < 1. \quad (27)$$

*Proof.* Taking into account relation (6), it suffices to prove the statement for  $m = 1 - e^{4i\theta}$  for  $\theta \in [-\pi/4, 0]$ . Applying the descending Landen transformation [1, Eq. 16.12.2] on  $\operatorname{sn}(K(1 - e^{4i\theta})u \mid 1 - e^{4i\theta})$  we obtain

$$\operatorname{sn}(K(1 - e^{4i\theta})u \mid 1 - e^{4i\theta}) = \frac{e^{-i\theta}}{\cos \theta} \frac{\operatorname{sn}(K'(e^{4i\theta})v \mid -\tan^2 \theta)}{1 - i \tan(\theta) \operatorname{sn}^2(K'(e^{4i\theta})v \mid -\tan^2 \theta)},$$

where  $v = ue^{i\theta} \cos \theta$ . Next, by applying transform [1, Eq. 16.10.2] together with equation

$$K'(e^{4i\theta}) = e^{-i\theta} K(\sin^2 \theta) \quad \text{for } \theta \in [0, \pi/4), \quad (28)$$

see [10] and references therein, one gets

$$\operatorname{sn}(K(1 - e^{4i\theta})u \mid 1 - e^{4i\theta}) = e^{-i\theta} \frac{\operatorname{sd}(K(\sin^2 \theta)u \mid \sin^2 \theta)}{1 - i \sin \theta \cos \theta \operatorname{sd}^2(K(\sin^2 \theta)u \mid \sin^2 \theta)}$$

which holds true for  $\theta \in [-\pi/4, 0]$ .

Further, by using identities for double arguments [1, Sec. 16.18], one verifies that

$$|\operatorname{sn}(K(1 - e^{4i\theta})u \mid 1 - e^{4i\theta})|^4 = \frac{\operatorname{sn}^2(2K(\sigma)u \mid \sigma)}{4 \operatorname{dn}^2(2K(\sigma)u \mid \sigma)} \frac{1 - \operatorname{cn}(2K(\sigma)u \mid \sigma)}{1 + \operatorname{cn}(2K(\sigma)u \mid \sigma)}$$

where  $\sigma := \sin^2 \theta$ . Consequently,

$$|\operatorname{sn}(K(1 - e^{4i\theta})u \mid 1 - e^{4i\theta})| < 1$$

if and only if

$$4d^2(1 + c) - s^2(1 - c) > 0$$

where the abbreviations  $s = \operatorname{sn}(2K(\sigma)u \mid \sigma)$ , etc., are used. This is, however, true for all  $u \in (0, 1)$  by Lemma 5, since  $\sigma \in [0, 1/2]$  for  $\theta \in [-\pi/4, 0]$ .  $\square$

**3.3. Region  $m \notin \mathbb{D}$ .** We take a closer look to the values of  $\operatorname{sn}(K(m)u \mid m)$  for  $|m| > 1$ . If in addition  $m \in \mathbb{C}_+$ , then, by (2), one has

$$K(m) = \mu^{1/2} [K(\mu) + iK'(\mu)] \quad (29)$$

where  $\mu = m^{-1}$ . The last formula and the Jacobi's real transformation [1, Eq. 16.11.2] yield

$$\operatorname{sn}(K(m)u \mid m) = \mu^{1/2} \operatorname{sn}([K(\mu) + iK'(\mu)]u \mid \mu).$$

Furthermore, by the addition formula [1, Eq. 16.17.1] and Jacobi's imaginary transformation [1, Sec. 16.20], one arrives at the expression

$$\operatorname{sn}(K(m)u \mid m) = \mu^{1/2} \frac{s \cdot d_1 + ic \cdot d \cdot s_1 \cdot c_1}{1 - d^2 \cdot s_1^2} \quad (30)$$

where  $s = \operatorname{sn}(K(\mu)u \mid \mu)$ ,  $s_1 = \operatorname{sn}(K(\mu_1)u \mid \mu_1)$ , etc., and  $\mu_1 = 1 - \mu$ . Identity (30) holds for  $m \in \mathbb{C}_+$ . If  $m \in \mathbb{C}_-$ , one has to change the sign at  $i$  in the nominator in the RHS of (30).

For  $m < 1$ , the function  $\operatorname{sn}(K(m)u \mid m)$  is real-valued. The interval  $(1, \infty)$  is a branch cut of  $\operatorname{sn}(K(m)u \mid m)$  and the limiting values for  $m > 1$  are

$$\operatorname{sn}(K(m)u \mid m) = \mu^{1/2} \frac{s \cdot d_1 \pm ic \cdot d \cdot s_1 \cdot c_1}{1 - d^2 \cdot s_1^2}$$

with the same abbreviation as above. The upper sign applies, if  $m$  is approached from the upper-plane  $\mathbb{C}_+$ , while the lower sign applies, if  $m$  is approached from the lower-plane  $\mathbb{C}_-$ . Note, however, that the absolute value of  $\operatorname{sn}(K(m)u \mid m)$  extends continuously on  $(1, m)$  and one has

$$|\operatorname{sn}(K(m)u \mid m)|^2 = \mu \frac{s^2 \cdot d_1^2 + c^2 \cdot d^2 \cdot s_1^2 \cdot c_1^2}{(1 - d^2 \cdot s_1^2)^2} \quad \text{for } m > 1. \quad (31)$$

All in all, one concludes the function  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  extends continuously on the whole plane  $\mathbb{C}$ . In addition to the continuity of  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$ , we will show that this function is also bounded.

**Proposition 7.** *For  $u \in (0, 1)$ , it holds*

$$\lim_{\substack{m \rightarrow \infty \\ m \in \mathbb{C} \setminus [1, \infty)}} \operatorname{sn}(K(m)u \mid m) = 0.$$

*In particular, function  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  is bounded on  $\mathbb{C} \setminus [1, \infty)$ .*

*Proof.* We will prove that  $\operatorname{sn}(K(m)u \mid m) \rightarrow 0$  for  $m \rightarrow \infty$  and  $\operatorname{Im} m > 0$ ; by the reflection relation (6), one obtains the same result for  $m \rightarrow \infty$  and  $\operatorname{Im} m < 0$ . Then the statement follows by the analyticity of  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  on  $\mathbb{C} \setminus [1, \infty)$ .

Let  $\operatorname{Im} m > 0$ . If  $m \rightarrow \infty$ , then  $\mu = m^{-1} \rightarrow 0$  and  $\mu_1 \rightarrow 1$ . By using formula (30) and applying the asymptotic expansions (7)–(9) together with (10)–(12), one obtains

$$\operatorname{sn}(K(m)u \mid m) = ie^{-i\pi u/2} 2^{2u-1} m^{(u-1)/2} (1 + o(1)), \quad \text{for } m \rightarrow \infty.$$

By taking into account the assumption  $u \in (0, 1)$ , one verifies the statement.  $\square$

Finally, we investigate the region  $m \in (1, \infty)$ . To do this, an auxiliary result is needed.

**Lemma 8.** *For any  $u \in (0, 1)$ , the function*

$$\varphi(u, \mu) := \operatorname{dc}^2(K(\mu_1)u \mid \mu_1) - \operatorname{dc}^2(K(\mu)u \mid \mu)$$

*is strictly increasing in  $\mu$  on  $(0, 1)$ . In addition, if  $u \in (0, 1/2]$ , then  $\varphi(u, \mu) < 1$  for all  $\mu \in (0, 1)$ , while if  $u \in (1/2, 1)$ , then there exists a unique  $\tilde{\mu} \in (1/2, 1)$  such that  $\varphi(u, \tilde{\mu}) = 1$ .*

*Proof.* We used the abbreviated notations  $s = \operatorname{sn}(K(\mu)u \mid \mu)$ ,  $s_1 = \operatorname{sn}(K(\mu_1)u \mid \mu_1)$ , etc., and also  $z = \operatorname{zn}(K(\mu)u \mid \mu)$  which is the Jacobi's zeta function, see [11, Sec. 3.6].

By using formulas [4, Eqs. 710.00 and 710.60], one obtains

$$\frac{d}{d\mu} \operatorname{dc}(K(\mu)u \mid \mu) = -\frac{s \cdot z}{2\mu c^2}. \quad (32)$$

Then (32) leads to

$$\frac{\partial}{\partial \mu} \varphi(u, \mu) = \frac{d_1 \cdot s_1 \cdot z_1}{\mu_1 c_1^3} + \frac{d \cdot s \cdot z}{\mu c^3} > 0$$

for all  $\mu \in (0, 1)$  and  $u \in (0, 1)$  since the Jacobi's zeta function  $\text{zn}(u \mid \mu)$  has positive values for  $u \in (0, K)$ , see [7]. Consequently, the function  $\varphi(u, \cdot)$  is strictly increasing on  $(0, 1)$  for any  $u \in (0, 1)$ .

The asymptotic formulas (7)–(12) yield

$$\lim_{\mu \rightarrow 1-} \varphi(u, \mu) = \tan^2(\pi u/2).$$

Thus, for  $u \in (0, 1/2]$ ,  $\varphi(u, \mu) < 1$  for all  $\mu \in (0, 1)$ . While, for  $u \in (1/2, 1)$ , there exists a unique solution  $\tilde{\mu} \in (0, 1)$  of the equation  $\varphi(u, \mu) = 1$ . Since clearly  $\varphi(u, 1/2) = 0$ ,  $\varphi(u, \mu) \leq 0$  for  $\mu \leq 1/2$  and hence  $\tilde{\mu} > 1/2$ .  $\square$

**Proposition 9.** *The following statements hold true.*

- i) For all  $(u, m) \in (0, 1/2] \times (1, \infty) \cup (0, 1) \times [2, \infty)$ , we have

$$|\text{sn}(K(m)u \mid m)| < 1.$$

- ii) Let  $u \in (1/2, 1)$  and  $\tilde{m} = \tilde{m}(u) \in (1, 2)$  be the unique solution of the equation

$$\text{dc}^2(K(1 - m^{-1})u \mid 1 - m^{-1}) - \text{dc}^2(K(m^{-1})u \mid m^{-1}) = 1.$$

Then, for all  $(u, m) \in (1/2, 1) \times (1, \tilde{m})$ , we have

$$|\text{sn}(K(m)u \mid m)| > 1,$$

while, for all  $(u, m) \in (1/2, 1) \times (\tilde{m}, 2)$ , we have

$$|\text{sn}(K(m)u \mid m)| < 1.$$

*Proof.* We use the abbreviated notations  $s = \text{sn}(K(\mu)u \mid \mu)$ ,  $s_1 = \text{sn}(K(\mu_1)u \mid \mu_1)$ , etc., as in the proof of Lemma 8. Recall also that we denoted  $\mu = m^{-1}$ .

First, we rewrite identity (31). By writing  $\mu s^2 = 1 - d^2$ ,  $d_1^2 = 1 - \mu_1 s_1^2$ ,  $\mu c^2 = d^2 - \mu_1$ , and finally  $c_1^2 = 1 - s_1^2$ , one verifies that

$$\mu(s^2 d_1^2 + c^2 d^2 s_1^2 c_1^2) = (1 - d^2 \cdot s_1^2)(1 - d^2 \cdot s_1^2 - d^2(1 - s_1^2) + (d^2 - \mu_1)s_1^2).$$

Consequently, from identity (31), one has

$$|\text{sn}(K(m)u \mid m)|^2 = 1 - \frac{d^2(1 - s_1^2) - (d^2 - \mu_1)s_1^2}{1 - d^2 \cdot s_1^2} = 1 - \frac{d^2 \cdot c_1^2 - \mu c^2 \cdot s_1^2}{1 - d^2 \cdot s_1^2}.$$

Next, by using [1, Eq. 16. 9. 3], the last identity can be further rewritten into the form

$$|\text{sn}(K(m)u \mid m)|^2 = 1 - \frac{c^2 \cdot c_1^2}{1 - d^2 \cdot s_1^2} (1 - \varphi(u, \mu)) \quad (33)$$

where the notation of Lemma 8 is used. Noticing that the factor

$$\frac{c^2 \cdot c_1^2}{1 - d^2 \cdot s_1^2} > 0$$

for all  $(m, \mu) \in (0, 1)^2$ , it follows from the formula (33) that the statement can be obtained by inspection of the values of  $\varphi(u, \mu)$ . This, however, is treated in Lemma 8. Indeed, by Lemma 8, for all  $(u, \mu) \in (0, 1/2] \times (0, 1) \cup (0, 1) \times (0, 1/2]$ , it holds that  $\varphi(u, \mu) < 1$ , so (33) then implies the validity of claim (i).

On the other hand, if  $u \in (1/2, 1)$ , according to Lemma 8, there exists a unique  $\tilde{\mu} \in (1/2, 1)$  such that  $\varphi(u, \mu) = 1$ . Hence by the monotonicity of  $\varphi(u, \cdot)$ ,  $\varphi(u, \mu) < 1$  for  $\mu < \tilde{\mu}$  and  $\varphi(u, \mu) > 1$  for  $\mu > \tilde{\mu}$ . This yields claim (ii) with  $\tilde{m} = \tilde{\mu}^{-1}$ .  $\square$

*Remark 10.* By applying [1, Eq. 16. 9. 3], identity (33) can be further simplified to the form

$$|\text{sn}(K(m)u \mid m)|^2 = 1 + \frac{\mu \text{sc}^2(K(\mu_1)u \mid \mu_1) - \text{dc}^2(K(\mu)u \mid \mu)}{1 + \text{sc}^2(K(\mu)u \mid \mu) \text{dc}^2(K(\mu_1)u \mid \mu_1)}.$$

This formula could be useful for more subtle investigation of properties of  $|\text{sn}(K(m)u \mid m)|$  for  $m > 1$ . First of all, it follows from claim (ii) of Proposition 9 that the function  $m \mapsto |\text{sn}(K(m)u \mid m)|$ , with  $u \in (1/2, 1)$ , has a maximum located in  $(1, \tilde{m})$ . However, it is not clear whether this maximum is unique, although there is a strong numerical evidence for this statement.

Now everything is prepared to prove the main theorem.



**3.4. Proof of Theorem 1.** Proof of assertion (i): Let  $u \in (0, 1)$ . We know already that

$$|\operatorname{sn}(K(m)u \mid m)| < 1, \quad (34)$$

for all  $m \in \overline{\mathbb{D}} \setminus \{1\} \cup \partial(\mathbb{D} \cup \mathbb{D}_1) \cup (2, \infty)$ , see Proposition 4, Lemma 6 and Proposition 9, part (ii).

Since  $\partial(\mathbb{D} \cup \mathbb{D}_1)$  is a compact set and taking into account that Proposition 7 together with the continuity of  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  implies  $|\operatorname{sn}(K(m)u \mid m)| \rightarrow 0$  as  $m \rightarrow \infty$ , there exists a constant  $c < 1$  such that, for all  $m \in \partial(\mathbb{D} \cup \mathbb{D}_1) \cup (2, \infty)$ ,

$$|\operatorname{sn}(K(m)u \mid m)| \leq c. \quad (35)$$

Finally, since  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  is analytic in the region  $\mathbb{C} \setminus (\overline{\mathbb{D} \cup \mathbb{D}_1} \cup (2, \infty))$ , continuous to the boundary and bounded in this region, the maximum modulus principle for unbounded regions, see, for example, [3, Thm. 15.1], implies that (35) holds for all  $m \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{D}_1)$ .

Proof of assertion (ii): Let  $u \in (0, 1/2]$ . By (10),  $|\operatorname{sn}(K(1)u \mid 1)| = 1$ . Further, according to Proposition 9, part (i), and Lemmas 3 and 6, we know that (34) holds for all  $m \in \partial((\mathbb{D}_1 \setminus \mathbb{D}) \cap \mathbb{C}_+) \setminus \{1\}$ . Since  $m \mapsto \operatorname{sn}(K(m)u \mid m)$  is analytic in  $(\mathbb{D}_1 \setminus \mathbb{D}) \cap \mathbb{C}_+$  and continuous to the boundary, the maximum modulus principle implies that, for all  $m \in (\mathbb{D}_1 \setminus \mathbb{D}) \cap \mathbb{C}_+$ ,

$$|\operatorname{sn}(K(m)u \mid m)| \leq 1.$$

However, the last inequality is strict, which can be verified by a standard procedure based on the averaging property of analytic functions, i.e., a function  $f$  analytic in a region  $G \subset \mathbb{C}$  satisfies

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

for any  $z_0$  and  $r > 0$  such that  $\{z \in \mathbb{C} \mid |z - z_0| \leq r\} \subset G$ . The rest now follows from (6) and the already proved assertion (i).

Proof of assertion (iii): Let  $u \in (1/2, 1)$ . By Proposition 9 part (iii), we know that there is a global maximum of the function  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  restricted to interval  $[1, 2]$  located at some point  $m^* \in (1, 2)$  and  $|\operatorname{sn}(K(m^*)u \mid m^*)| > 1$ . Since for all  $m \in (\partial(\mathbb{D}_1 \setminus \mathbb{D})) \cap \mathbb{C}_+$  we already known that  $|\operatorname{sn}(K(m)u \mid m)| < 1$ . Thus, by the maximum modulus principle,  $m^*$  is the global maximum of  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  restricted to  $(\mathbb{D}_1 \setminus \mathbb{D}) \cap \mathbb{C}_+$ . Now it suffices to take into account (6) and the already proved assertion (i).  $\square$

#### 4. NUMERICAL COMPUTATIONS AND CONCLUDING REMARKS

Theorem 1 shows that the extremal properties of  $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$  are interesting particularly in the region  $\mathbb{D}_1 \setminus \mathbb{D}$  and the interval  $[1, 2]$ ; recall we assume  $u \in (0, 1)$ . To simplify notations, we denote

$$\sigma(u, m) := |\operatorname{sn}(K(m)u \mid m)|. \quad (36)$$

For different values of  $u \in (0, 1)$ , the region in  $\mathbb{C}$  where  $\sigma(u, \cdot) \geq 1$  is plotted in Figure 1. As numerics suggests, there is exactly one maximum of  $\sigma(u, \cdot)$  at  $m^*(u) \in [1, 2]$  and its position is indicated in Figure 1 as well. Moreover, the position  $m^*(u)$  of the maximum and the

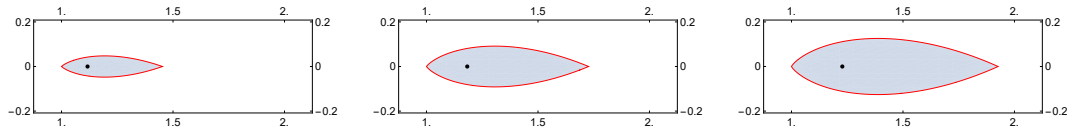


FIGURE 1. The region in  $\mathbb{C}$  where  $\sigma(u, \cdot) \geq 1$  for  $u = 0.7$ ,  $u = 0.8$  and  $u = 0.9$ . Black ball shows the position  $m^*$  of the maximum of  $\sigma(u, \cdot)$ .

maximal value of  $\sigma(u, \cdot)$  are plotted in Figure 2. The maximal values exceed 1 only very little; the value of the global maximum of  $\sigma(u, m)$  on  $[0, 1] \times \mathbb{C}$  reads approx. 1.01038 and it is located at approx.  $(u, m) = (0.69098, 1.11015)$ . Finally, it even seems that  $\sigma(u, \cdot)$  is convex in  $(1, 2)$  for  $u \in (0, 1/2]$  and concave for  $u \in (1/2, 1)$ , see Figure 3.

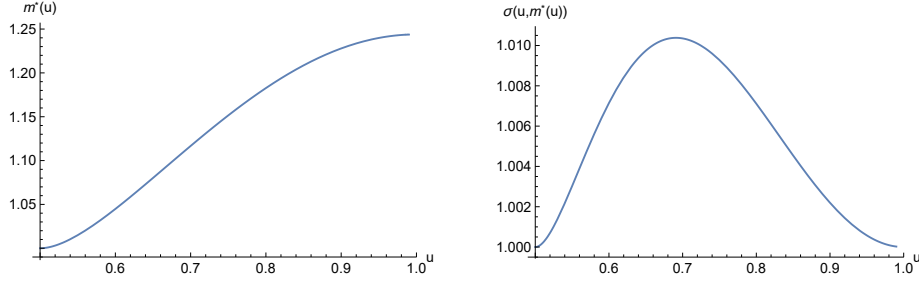


FIGURE 2. The position  $m^*(u)$  of the maximum of  $\sigma(u, m)$  (left) and the maximal value  $\sigma(u, m^*(u))$  (right).

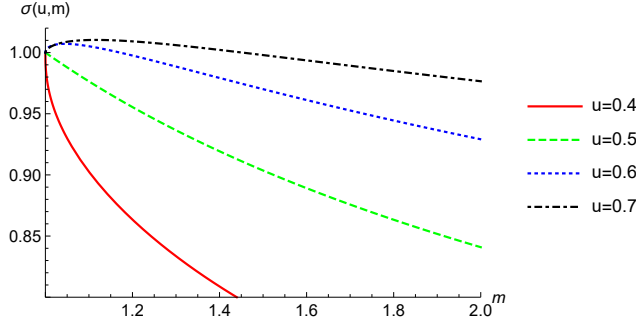


FIGURE 3. The plots of  $\sigma(u, \cdot)$  for  $u = 0.4$  (red, solid),  $u = 0.5$  (green, dashed),  $u = 0.6$  (blue, dotted) and  $u = 0.7$  (black, dot-dashed).

## 5. APPLICATIONS IN LAPLACE-TYPE INTEGRAL AND SPECTRA OF JACOBI MATRICES

Although the properties of the function  $m \mapsto \text{sn}(K(m)u \mid m)$  are of independent interest, our study was motivated by the spectral analysis of a one-parameter family of operators in  $\ell^2(\mathbb{N})$  associated with semi-infinite, in general non-symmetric, Jacobi matrices:

$$J(k) = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 2k & & \\ & 2k & 0 & 3 & \\ & & 3 & 0 & 4k \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad k \in \overline{\mathbb{D}}, \quad (37)$$

see [14]. The case  $k \notin \overline{\mathbb{D}}$  can be converted to a similar problem by considering  $k^{-1}J(k)$ . The operator  $J(k)$  turns out to be strongly linked with Jacobian elliptic functions, in particular, the parameter  $k$  indeed coincides with the modulus, i.e.  $m = k^2$ . Notice that with our restriction on values of  $k$ , we have  $m \in \overline{\mathbb{D}}$ , thus we advantageously avoid the region where it is possible that  $|\text{sn}(K(m)u \mid m)| > 1$ .

While, for  $k \in \mathbb{D}$ , the spectrum of  $J(k)$  is the discrete set  $(2\mathbb{Z} + 1)\pi/(2K(m))$ , a sudden change of spectral character, sometimes called spectral phase transition, is observed when  $k$  reaches  $\partial\mathbb{D}$ . Namely, for  $k$  in the latter set, the spectrum of  $J(k)$  is purely essential and, if in addition  $k \neq \pm 1$ , the spectrum coincides with the whole  $\mathbb{C}$ . The special self-adjoint and explicitly diagonalizable case  $J(\pm 1)$  is omitted here, see [14] for details.

The essential ingredient for the spectral results above is the analysis of functions

$$\begin{aligned} C_l(z; m) &:= \int_0^{2K(m)} e^{-zt} \text{cn}(t \mid m) \text{sn}^l(t \mid m) dt, \\ D_l(z; m) &:= \int_0^{2K(m)} e^{-zt} \text{dn}(t \mid m) \text{sn}^l(t \mid m) dt, \end{aligned} \quad (38)$$

where  $l \in \mathbb{N}_0$ ,  $z \in \mathbb{C}$ ,  $m \in \overline{\mathbb{D}} \setminus \{1\}$ , and the integration is carried out through the line segment in  $\mathbb{C}$  connecting 0 and  $2K(m)$ . The reason is that the sequence  $v = v(z; m)$  with entries

$$v_{2l+1} := i(-1)^l m^{l/2} e^{iK(m)z} C_{2l}(iz; m), \quad v_{2l+2} := (-1)^{l+1} m^{l/2} e^{iK(m)z} D_{2l+1}(iz; m), \quad (39)$$

satisfies an infinite system of difference equations, compactly written as

$$(J(k) - z)v = -2 \cos(K(m)z) e_1, \quad (40)$$

where  $e_1$  denotes the first vector of the standard basis of  $\ell^2(\mathbb{N})$ .

It follows from (40) that  $z \in (2\mathbb{Z} + 1)\pi/(2K(m))$  are eigenvalues of  $J(k)$  if the corresponding  $v(z; m)$  belongs to  $\ell^2(\mathbb{N})$ . However, the latter condition is satisfied if  $k \in \mathbb{D}$  due to the factor  $m^{l/2}$  in (39) and the fact that the integrals in (38) are majorized by a constant independent on  $l$ . Indeed, by Theorem 1,  $|\operatorname{sn}(u | m)| \leq 1$  for all  $m = k^2 \in \mathbb{D}$  and  $u \in (0, 2K(m))$ .

The case  $k \in \partial\mathbb{D} \setminus \{\pm 1\}$  is more delicate. One works with singular sequences instead of eigenvectors and it is crucial to analyze the asymptotic behavior of  $v_n$ , for  $n \rightarrow \infty$ , leading to asymptotic expansion of integrals

$$I_l(f) = \int_0^{2K(m)} f(t) \operatorname{sn}^l(t | m) dt, \quad l \rightarrow \infty, \quad (41)$$

where  $f$  is a function analytic on a neighborhood of  $[0, 2K(m)]$ . Note that, for special choices of  $f$ , (41) becomes (38). The integral (41) is in the form suitable for application of the saddle point method, see, e.g., [9, p. 417, Thm. 1.1] and [19, Sec. II.4] for details. Nevertheless, we need to know maximal values of  $|\operatorname{sn}(\cdot | m)|$  on  $(0, 2K(m))$ . Thus, Theorem 1 enters again, now for  $m \in \partial\mathbb{D} \setminus \{1\}$ , and we deduce that the function  $|\operatorname{sn}(\cdot | m)|$  restricted to  $(0, 2K(m))$  has a unique global maximum located at  $u = K(m)$  and  $\operatorname{sn}(K(m) | m) = 1$ . Then the straightforward application of the saddle point method yields that, for all  $m \in \overline{\mathbb{D}} \setminus \{1\}$ ,

$$I_l(f) = \frac{\sqrt{2\pi}}{2^r r!} \frac{f^{(2r)}(K(m))}{(1-m)^{r+1/2}} l^{-r-1/2} + O(l^{-r-1}), \quad l \rightarrow \infty,$$

where  $r \in \mathbb{N}_0$  is the lowest integer such that  $f^{(2r)}(K(m)) \neq 0$ . In particular, one gets

$$C_l(z; m) = \frac{\sqrt{2\pi} z}{(1-m)e^{K(m)z}} \frac{1}{l^{3/2}} + O(l^{-2}), \quad D_l(z; m) = \frac{\sqrt{2\pi}}{e^{K(m)z}} \frac{1}{l^{1/2}} + O(l^{-1}), \quad l \rightarrow \infty.$$

For more details and connected spectral results on  $J(k)$ , see [14].

#### ACKNOWLEDGEMENTS

The research of P. S. is supported by the *Swiss National Foundation*, SNF Ambizione grant No. PZ00P2\_154786. F. Š. gratefully acknowledges the kind hospitality of the Mathematisches Institut, Universität Bern; his research was also supported by grant No. GA13-11058S of the Czech Science Foundation.

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